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# A SEMIPARAMETRIC FAMILY OF BIVARIATE COPULAS: DEPENDENCE PROPERTIES AND ESTIMATION PROCEDURES

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Joint work with Cécile Amblard.

## Outline

1. Definition and basic properties.
2. First sub-family, the case  $\theta(1) = 0$ .
3. Second sub-family, the case  $\phi(1) = 0$ .
4. Inference procedures.
5. Simulation results.
6. Real data.

## 1. Definition and basic properties.

**Definition.** Let  $I$  be the unit interval. The family is defined for all  $(u, v) \in I^2$  by,

$$C_{\theta, \phi}(u, v) = uv + \theta[\max(u, v)]\phi(u)\phi(v).$$

where  $\phi$  and  $\theta$  are differentiable  $I \rightarrow \mathbb{R}$  functions (vanishing at most on isolated points).

**Theorem.**  $C_{\theta, \phi}$  is a copula if and only if  $\phi$  and  $\theta$  satisfy the following conditions:

- boundary conditions:  $\phi(0) = 0$  and  $(\phi\theta)(1) = 0$ ,
- $\theta$  is non increasing on  $I$ ,
- $\phi'(u)(\theta\phi)'(v) \geq -1$  for all  $0 \leq u \leq v \leq 1$ .

**Remark.** The family can be split in two sub-families according to  $\theta(1) = 0$  or  $\phi(1) = 0$ .

## Measure of association.

Let  $(X, Y)$  a random pair with joint distribution  $H(x, y) = C(F(x), G(y))$ . Spearman's Rho: probability of concordance minus the probability of discordance of two random pairs with respective joint cumulative law  $C(F, G)$  and  $FG$ .

$$\rho = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3.$$

In the case of  $C = C_{\theta, \phi}$ , we have

$$\rho_{\theta, \phi} = 12 \left[ \Phi^2(1)\theta(1) - \int_0^1 \Phi^2(t)\theta'(t)dt \right],$$

where  $\Phi(t) = \int_0^t \phi(u)du$ .

### Remark.

- If  $\theta(1) = 0$ , then  $\rho_{\theta, \phi} \geq 0$ .
- If  $\theta$  is a constant function, then  $\rho_{\theta, \phi} = 12\theta\Phi^2(1)$ .

## Upper tail dependence.

The upper tail dependence coefficient is defined as

$$\lambda = \lim_{t \rightarrow 1} \mathbb{P}(F(X) > t | G(Y) > t) = \lim_{u \rightarrow 1} \frac{\bar{C}(u, u)}{1 - u},$$

where  $\bar{C}$  is the survival copula, *i.e.*  $\bar{C}(u, v) = 1 - u - v + C(u, v)$ .

In the case where  $C = C_{\theta, \phi}$ , we have

$$\lambda_{\theta, \phi} = -\phi^2(1)\theta'(1).$$

### Remark.

- If  $\phi(1) = 0$ , then  $\lambda_{\theta, \phi} = 0$ .
- If  $\theta$  is a constant function, then  $\lambda_{\theta, \phi} = 0$ .

2. First sub-family, the case  $\theta(1) = 0$ .**Examples.**

- Fréchet upper bound. Choosing  $\phi(x) = x$  and  $\theta(x) = (1 - x)/x$  yields  $C_{\theta,\phi}(u, v) = M(u, v) = \min(u, v)$ .
- Independent copula.  $\theta(x) = 0$  yields  $C_{\theta,\phi}(u, v) = \Pi(u, v) = uv$ .
- Cuadras-Augé family:  $\phi(x) = x$  and  $\theta(x) = x^{-\alpha} - 1$ ,  $0 \leq \alpha \leq 1$  yields

$$C_{\theta,\phi}(u, v) = \min(u, v)^\alpha (uv)^{1-\alpha} = M^\alpha(u, v) \Pi^{1-\alpha}(u, v),$$

which is the weighted geometric mean of  $M$  and  $\Pi$ .

**Remark.**

- $\theta(1) = 0$  and  $\theta'(u) \leq 0$  imply  $\theta(u) \geq 0$  for all  $u \in I$ .
- $0 \leq \rho_{\theta,\phi} \leq 1 \longrightarrow$  Modelling of positive dependences.
- Lower (0) and upper bounds (1) of  $\rho_{\theta,\phi}$  and  $\lambda_{\theta,\phi}$  are reached respectively by the  $\Pi$  and  $M$  copulas.

## Dependence properties: definitions.

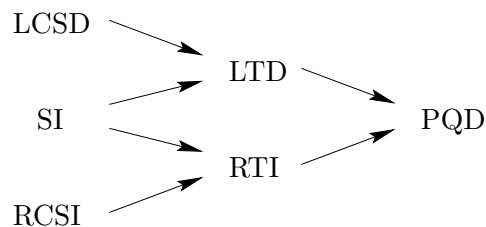
Assume  $X$  and  $Y$  are exchangeable.  $X$  and  $Y$  are

- Positively Quadrant Dependent (PQD) if  $\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$  for all  $(x, y)$ .
- Left Tail Decreasing (LTD) if  $\mathbb{P}(Y \leq y | X \leq x)$  is non-increasing in  $x$  for all  $y$ .
- Right Tail Increasing (RTI) if  $\mathbb{P}(Y > y | X > x)$  is nondecreasing in  $x$  for all  $y$ .
- Stochastically Increasing (SI) if  $\mathbb{P}(Y > y | X = x)$  is nondecreasing in  $x$  for all  $y$ .
- Left Corner Set Decreasing (LCSD) if  $\mathbb{P}(X \leq x, Y \leq y | X \leq x', Y \leq y')$  is non-increasing in  $x'$  and  $y'$  for all  $(x, y)$ .
- Right Corner Set Increasing (RCSI) if  $\mathbb{P}(X > x, Y > y | X > x', Y > y')$  is nondecreasing in  $x'$  and  $y'$  for all  $(x, y)$ .



**Theorem.**  $X$  and  $Y$  are:

- PQD iff  $\phi(u)$  has a constant sign on  $I$ .
- LTD or LCSD iff either  $\{\phi(u)/u$  is non increasing and  $\forall u \in I, \phi(u) \geq 0\}$  or  $\{\phi(u)/u$  is non decreasing and  $\forall u \in I, \phi(u) \leq 0\}$ .
- RTI or RCSI iff  $\phi(u)/(1-u)$  and  $\theta(u)\phi(u)/(1-u)$  are monotone.
- SI iff either  $\{\phi$  and  $\theta\phi$  are concave and  $\forall u \in I, \phi(u) \geq 0\}$  or  $\{\phi$  and  $\theta\phi$  are convex and  $\forall u \in I, \phi(u) \leq 0\}$ .



Implications in the general case



Implications in the sub-family

### 3. Second sub-family, the case $\phi(1) = 0$ .

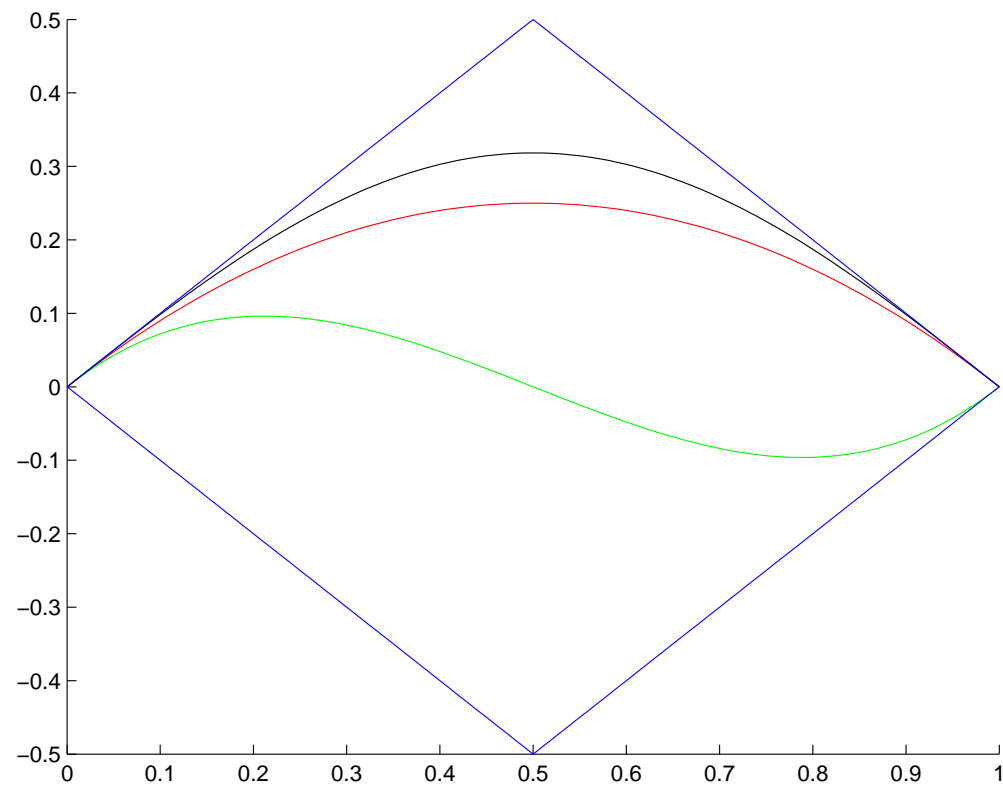
In this case, we restrict ourselves to a constant function  $\theta$ , *i.e.*  $\theta(x) = \theta \in [-1, 1]$ .

**Theorem.**  $C_{\theta, \phi}$  is a copula if and only if  $\phi$  and  $\theta$  satisfy the following conditions:

- boundary conditions:  $\phi(0) = 0$  and  $\phi(1) = 0$ ,
- $|\phi'(x)| \leq 1$  for all  $x \in I$ ,
- $|\phi(x)| \leq \min(x, 1 - x)$ , for all  $x \in I$ .

#### Examples.

- $\phi(x) = \min(x, 1 - x)$ : upper bound of the above theorem,
- $\phi(x) = x(1 - x)$ : Farlie-Gumbel-Morgenstern family of copulas (Morgenstern, 1956), which contains all copulas with both horizontal and vertical quadratic sections (Quesada-Molina, Rodríguez-Lallena, 1995)
- $\phi(x) = x(1 - x)(1 - 2x)$ : symmetric copulas with cubic sections (Nelsen *et al*, 1997),
- $\phi(x) = \pi^{-1} \sin(\pi x)$ .



Upper bound, Farlie-Gumbel-Morgenstern, cubic sections, sinus.

**Measure of association.** The Spearman's Rho can be rewritten as:

$$\rho_{\theta,\phi} = 12\theta \left( \int_I \phi(u) du \right)^2,$$

and it follows that  $-3/4 \leq \rho_{\theta,\phi} \leq 3/4$  for all  $\theta \in [-1, 1]$ . Similar bounds hold for the Kendall's Tau:  $-1/2 \leq \tau_{\theta,\phi} \leq 1/2$ .

**Upper tail dependence.**  $\rho_{\theta,\phi} = 0$ .

**Dependence properties.** Similar to the previous family in the case  $\theta > 0$ .

## Symmetry properties: definitions.

- $X$  is **symmetric** about  $a$  if  $(X - a)$  and  $(a - X)$  are identically distributed (id).
- $X$  and  $Y$  are **exchangeable** if  $(X, Y)$  and  $(Y, X)$  are id.
- $(X, Y)$  is **marginally symmetric** about  $(a, b)$  if  $X$  and  $Y$  are symmetric about  $a$  and  $b$  respectively.
- $(X, Y)$  is **radially symmetric** about  $(a, b)$  if  $(X - a, Y - b)$  and  $(a - X, b - Y)$  are id.
- $(X, Y)$  is **jointly symmetric** about  $(a, b)$  if the pairs  $(X - a, Y - b)$ ,  $(a - X, b - Y)$ ,  $(X - a, b - Y)$  and  $(a - X, Y - b)$  are id.

**Theorem.** In the  $C_{\theta, \phi}$  family:

- If  $X$  and  $Y$  are id then  $X$  and  $Y$  are exchangeable.

Besides, if  $(X, Y)$  is marginally symmetric about  $(a, b)$  then:

- $(X, Y)$  is radially symmetric about  $(a, b)$  if and only if  
either  $\forall u \in I, \phi(u) = \phi(1 - u)$  or  $\forall u \in I, \phi(u) = -\phi(1 - u)$ .
- $(X, Y)$  is jointly symmetric about  $(a, b)$  if and only if  $\forall u \in I, \phi(u) = -\phi(1 - u)$ .

## 4. Inference procedures.

### Assumptions.

- We restrict ourselves to the second sub-family, with constant function  $\theta$ :

$$C(u, v) = uv + \theta\phi(u)\phi(v).$$

→ Estimation of  $\theta$  (scalar) and  $\phi$  (univariate function).

→ Identifiability problem:  $(\theta, \phi)$  and  $(\alpha\theta, \phi/\sqrt{\alpha})$  yield the same copula for all  $\alpha > 0$ .

- We focus on the PQD case:  $\theta > 0$  and  $\phi$  has a constant sign.

Under these assumptions, the family can be rewritten

$$C(u, v) = uv + \psi(u)\psi(v),$$

where  $\psi(x) = \sqrt{\theta}|\phi(x)|$ .

→ The estimation of  $C$  reduces to the estimation of  $\psi$  (positive univariate function).

## Estimation of $\psi$

### 1) Preprocessing:

- $\{(x_i, y_i), i = 1, \dots, n\}$  a sample of  $(X, Y)$  from the cdf  $H(x, y) = C(F(x), G(y))$ .
- Rank transformations:  $u_i = \text{rank}(x_i)/n$  and  $v_i = \text{rank}(y_i)/n$ .  
 $\{(u_i, v_i), i = 1, \dots, n\}$  an approximate sample from the copula  $C(u, v)$ .
- Pseudo-observations  $\{w_i = \max(u_i, v_i), i = 1, \dots, n\}$  from  $C(w, w) = w^2 + \psi(w)$ .

### 2) Projection estimate: linear combination of basis functions: $\{e_k, k \geq 1\}$

$$\hat{\psi}(w) = \sum_{k \geq 1} a_k e_k(w), \quad w \in I.$$

Choice of the set of functions:

- no orthogonality condition,
- boundary conditions  $e_k(0) = e_k(1) = 0$  for all  $k \geq 1$  so that  $\hat{\psi}(0) = \hat{\psi}(1) = 0$ .

### 3) **Optimization problem:** Define

- $w_{1,n} \leq \dots \leq w_{n,n}$ , the ordered pseudo-observations,
- $M$  and  $M'$  two matrices  $M_{i,k} = e_k(w_{i,n})$ ,  $M'_{i,k} = e'_k(w_{i,n})$ ,  $k \geq 1$ ,  $i \in \{1, \dots, n\}$ ,
- $a$  and  $b$  two vectors  $b_i = (i/(n+1) - w_{i,n}^2)^{1/2}$ ,  $a_i$  unknown,  $i \in \{1, \dots, n\}$ .

Definition of the estimator.

- $\hat{\psi}(w_{i,n}) = C(w_{i,n}, w_{i,n}) - w_{i,n}^2 \simeq i/(n+1) - w_{i,n}^2$  for  $i = 1, \dots, n$  can be rewritten

$$\min_a \|Ma - b\|^2,$$

- $\hat{\psi}(w_{i,n}) \geq 0$  can be rewritten  $Ma \geq 0$ ,
- $|\hat{\psi}(w_{i,n})| \leq 1$  can be rewritten  $-1 \leq M'a \leq 1$ .

→ Constrained least-square problem.



## Estimation of the Spearman's rho

Recall that

$$\rho_{\theta,\phi} = 12\theta \left( \int_I \phi(u) du \right)^2 = 12 \left( \int_I \psi(u) du \right)^2.$$

Replacing  $\psi$  by  $\hat{\psi}$  yields the following semi-parametric estimator:

$$\hat{\rho}_{\text{SP}} = 12 \left( \sum_{k \geq 1} a_k \beta_k \right)^2,$$

where we have introduced  $\beta_k = \int_I e_k(u) du$ .

Another solution: adapt the nonparametric estimator of the Kendall's Tau introduced in (Genest, Rivest, 1993) to obtain

$$\hat{\rho}_{\text{NP}} = \frac{6}{n(n-1)} \sum_{i=1}^n \sum_{j=1}^n \mathbf{1}\{u_j < u_i, v_j < v_i\} - \frac{3}{2},$$

## Estimation of high probability regions

**Definition.** The  $\alpha$ -quantile of the copula  $C$  is defined by

$$Q_\alpha = \inf\{\lambda(S) : \mathbb{P}(S) \geq \alpha, S \subset I^2\}, \quad 0 < \alpha \leq 1,$$

where  $\lambda$  is the Lebesgue measure on  $I^2$ .

**Partitions.**  $\{I_k, k = 1, \dots, N\}$  be the equidistant  $N$ -partition of  $I$ ,  $K_{k,\ell} = I_k \times I_\ell$  the associated  $N \times N$  grid. Denote  $\delta_{k,\ell} \in \{0, 1\}$ ,  $(k, \ell) \in \{1, \dots, N\}^2$ .

**Estimator:**  $\hat{Q}_\alpha = \bigcup_{k,\ell} K_{k,\ell} \mathbf{1}\{\delta_{k,\ell} = 1\}$ .

**Optimization problem.** The  $\delta_{k,\ell}$  are defined by

$$\min \sum_{k=1}^N \sum_{\ell=1}^N \delta_{k,\ell},$$

under the constraints  $\delta_{k,\ell} \in \{0, 1\}$  and  $\sum_{k=1}^N \sum_{\ell=1}^N \delta_{k,\ell} \hat{P}(K_{k,\ell}) \geq \alpha$ ,

where  $\hat{P}(K_{k,\ell})$  is an estimation of the probability  $P(K_{k,\ell})$ .

**Algorithm.**

- First step: sort the  $\hat{P}(K_{k,\ell})$  in decreasing order to obtain the sequence  $\tilde{P}_\tau$ ,  $\tau = 1, \dots, N^2$ .
- Second step: Computation of the number of subsets of the partition:

$$J = \min \left\{ j, \sum_{\tau=1}^j \tilde{P}_\tau \geq \alpha \right\}.$$

- Third step: selection of the  $J$  first subsets:  $\delta_{k,\ell} = 1$  if  $1 \leq \tau(k, \ell) \leq J$ ,

**Estimation of  $P(K_{k,\ell})$ .** Two solutions:

- Semi-parametric estimate based on  $\hat{\psi}$

$$\hat{P}_{\text{sp}}(K_{k,\ell}) = \frac{1}{N^2} + \left( \hat{\psi}\left(\frac{k}{N}\right) - \hat{\psi}\left(\frac{k-1}{N}\right) \right) \left( \hat{\psi}\left(\frac{\ell}{N}\right) - \hat{\psi}\left(\frac{\ell-1}{N}\right) \right).$$

- Nonparametric estimate

$$\hat{P}_{\text{np}}(K_{k,\ell}) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{(u_i, v_i) \in K_{k,\ell}\}.$$

## 5. Simulation results.

Numerical experiments on the family of copulas  $C_k$  generated by the set of functions

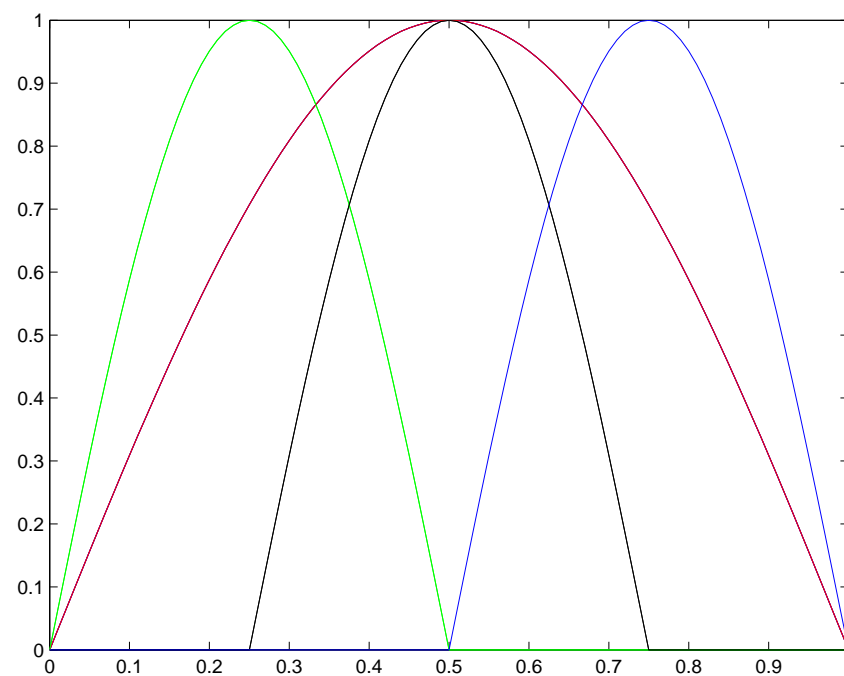
$$\forall k \geq 1, \quad \psi_k(x) = 1 - (x^k + (1-x)^k)^{1/k}, \quad x \in I.$$

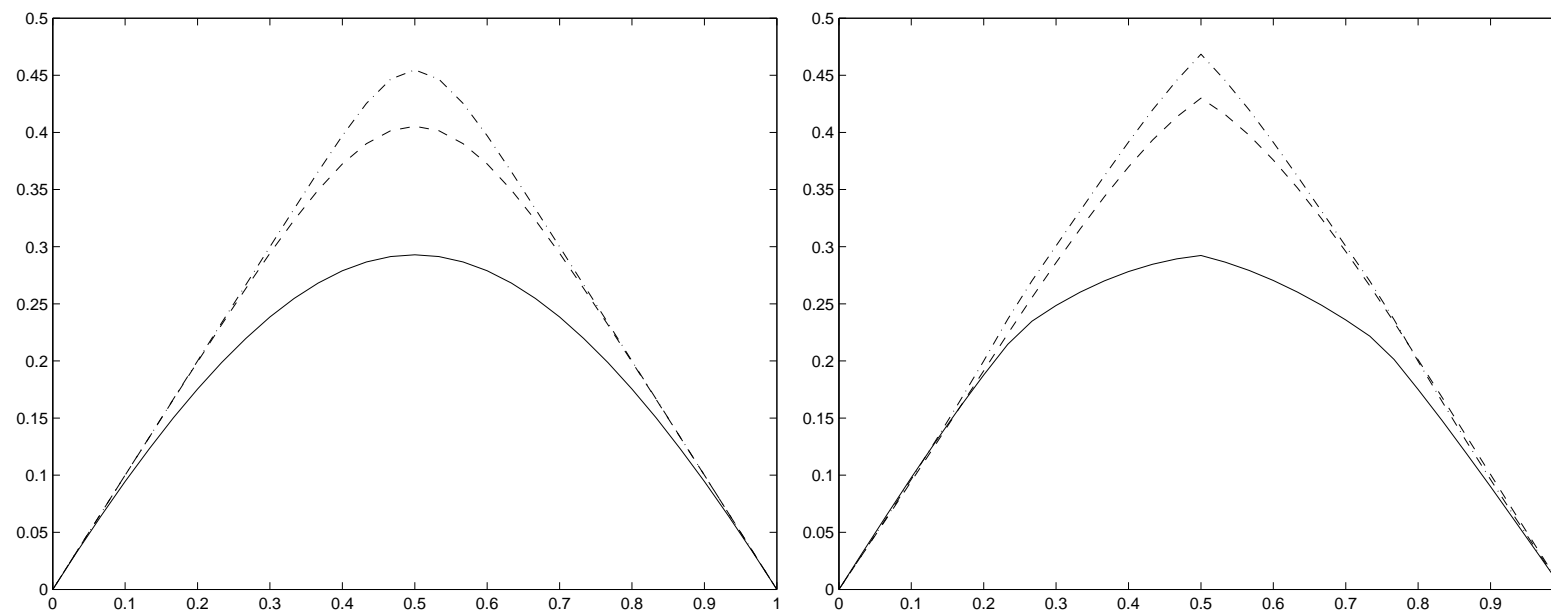
- When  $k = 1$ ,  $C_1$ : uniform distribution on  $I^2$ . Spearman's Rho  $\rho_1 = 0$ .
- When  $k \rightarrow \infty$ ,  $\psi_k(x) \rightarrow \psi_\infty(x) = \min(x, 1-x)$  for all  $x \in I$ .  
 $C_\infty$ : mixture of two uniform distributions on the squares  $[0, 1/2]^2$  and  $[1/2, 1]^2$  with mixing parameter  $1/2$ . Spearman's Rho  $\rho_\infty = 3/4$  (the maximum value in the sub-family).
- When  $1 < k < \infty$ , bivariate distribution “interpolating” between the two previous ones.

Chosen basis of functions:

$$e_{s,\ell}(x) = \sin\left(\frac{\pi}{2}(2^{s+1}x - \ell)\right) \mathbf{1}_{\{2^{s+1}x \in [\ell, \ell+2]\}},$$

$s$  is a scale parameter,  $\ell$  is a location parameter.

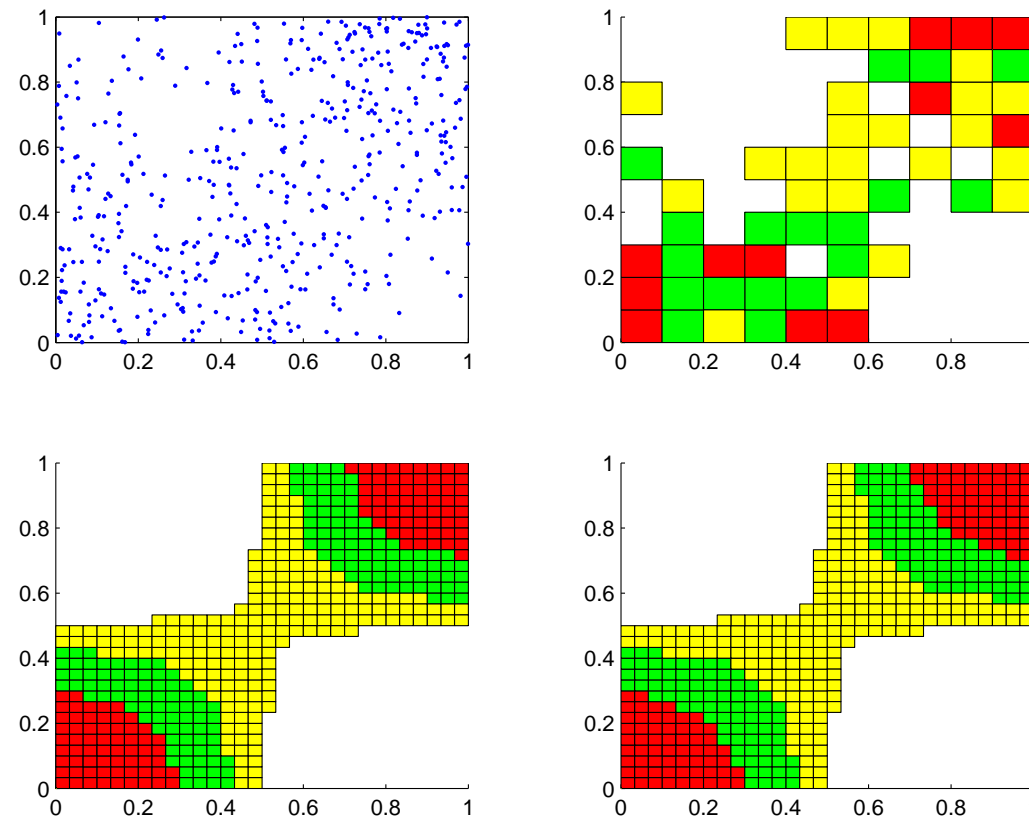




True functions  $\psi_k(x)$ ,  $k \in \{2, 4, 8\}$  – Estimated functions  $\hat{\psi}_k(x)$ ,  $k \in \{2, 4, 8\}$ ,  $n = 100$ .

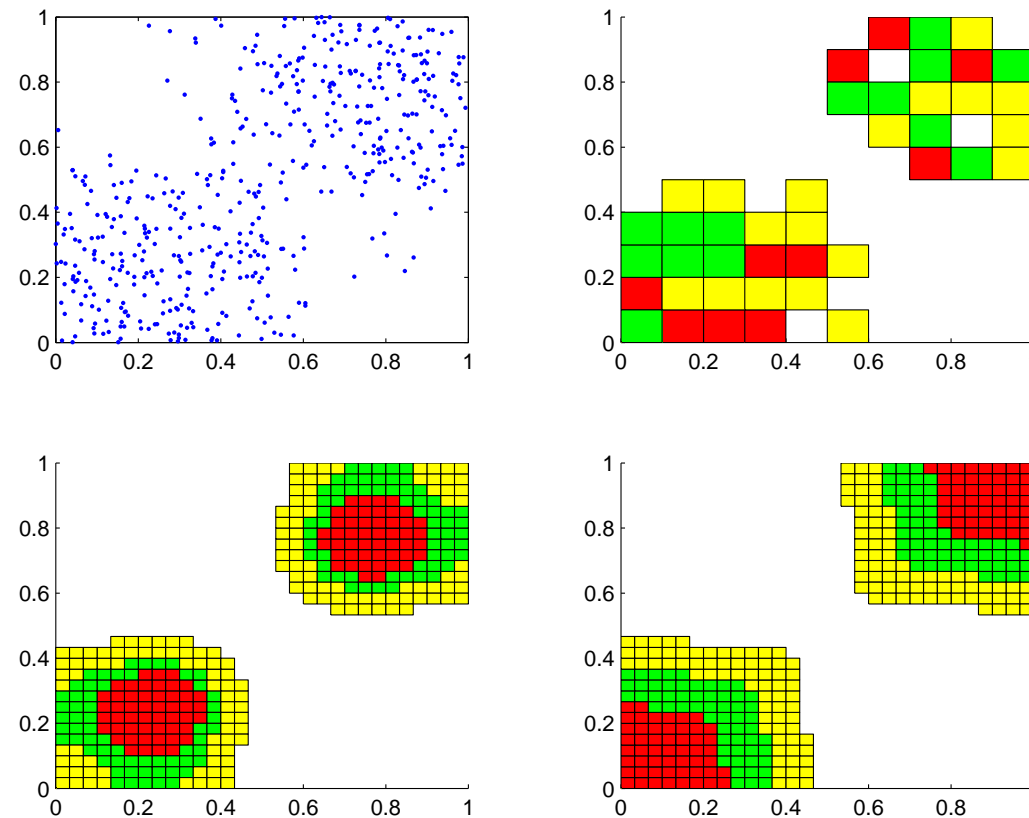
$k$	$\rho_k \times 10^{-2}$	$\text{mean}(\hat{\rho}_{\text{SP}}) \times 10^{-2}$	$\text{mean}(\hat{\rho}_{\text{NP}}) \times 10^{-2}$
1	0	0.81	0.18
2	42.5	43.0	41.2
4	66.4	65.8	64.3
6	71.2	70.6	68.8
8	72.8	72.1	70.2

Estimation of the generating function and of the Spearman's Rho ( $\rho_k$ ). The mean value of the estimates  $\hat{\rho}_{\text{SP}}$  and  $\hat{\rho}_{\text{NP}}$  are evaluated on 100 repetitions.

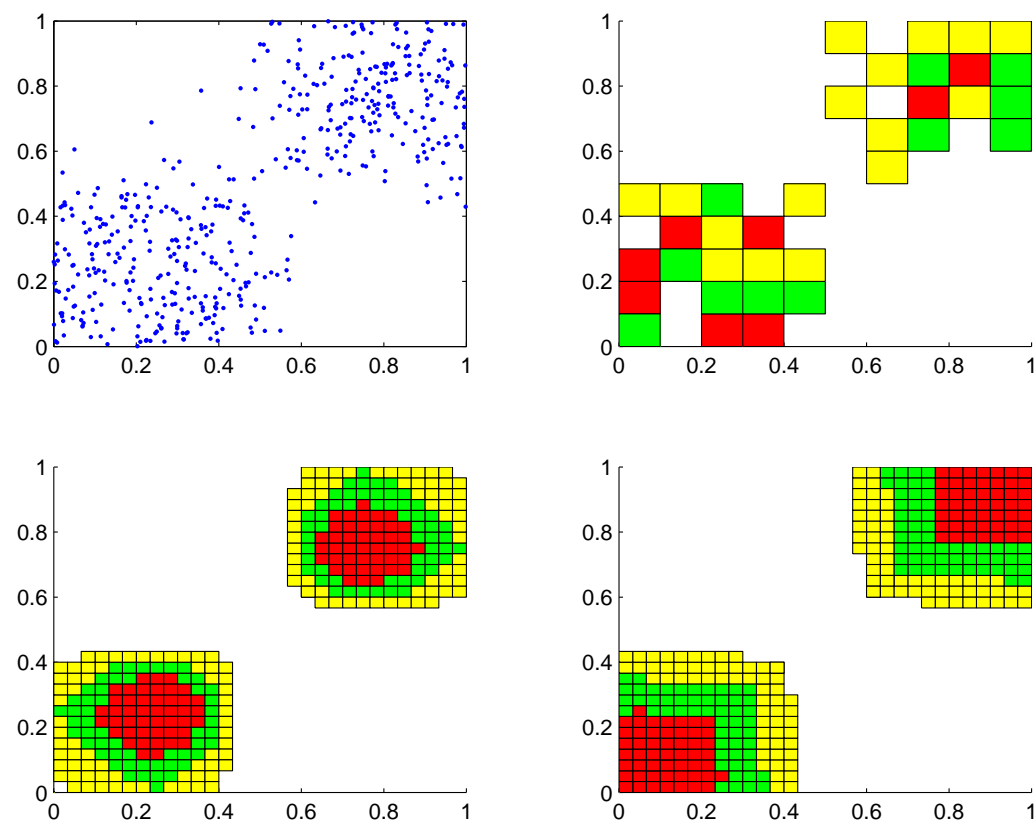


Estimation of high probability regions  $Q_\alpha$  from  $C_2$ . Red:  $\alpha = 0.25$ , green:  $\alpha = 0.5$ , yellow:  $\alpha = 0.75$ . Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function  $\psi$ , ( $n = 500$ ).





Estimation of high probability regions  $Q_\alpha$  from  $C_4$ . Red:  $\alpha = 0.25$ , green:  $\alpha = 0.5$ , yellow:  $\alpha = 0.75$ . Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function  $\psi$ , ( $n = 500$ ).

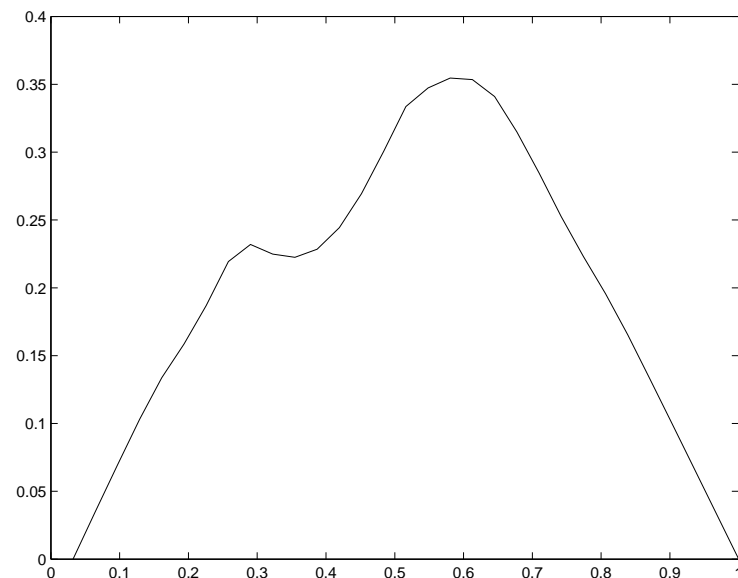


Estimation of high probability regions  $Q_\alpha$  from  $\mathcal{C}_8$ . Red:  $\alpha = 0.25$ , green:  $\alpha = 0.5$ , yellow:  $\alpha = 0.75$ . Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function  $\psi$ , ( $n = 500$ ).

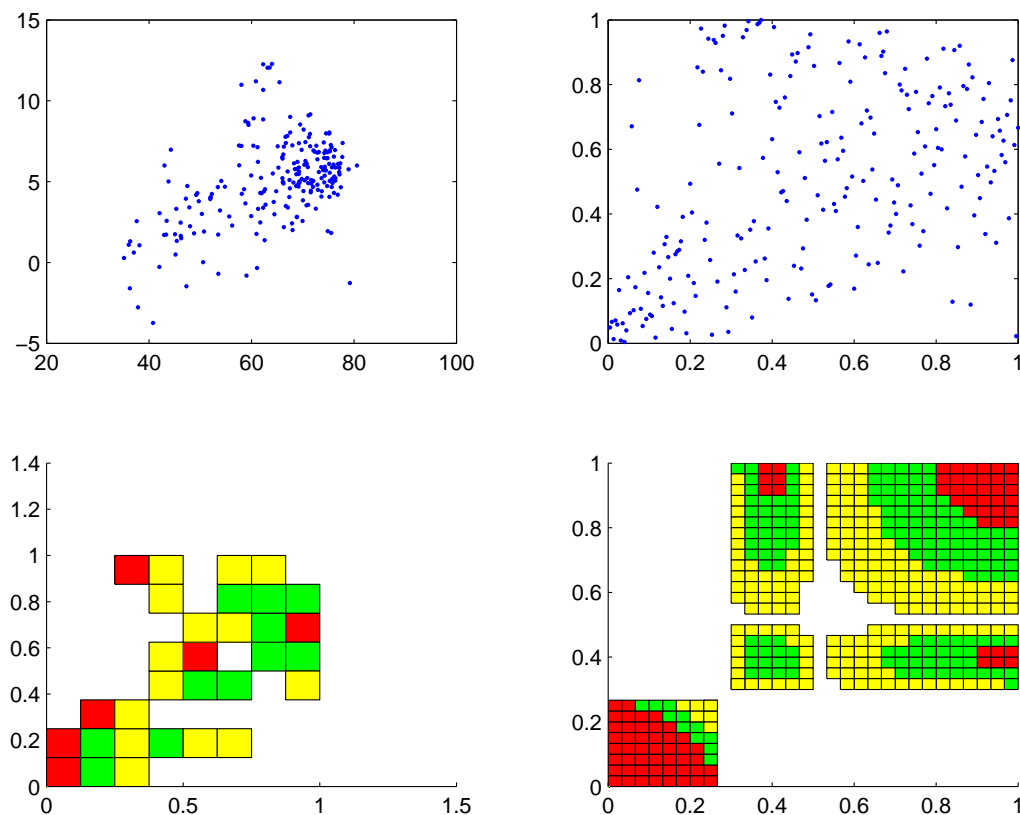
## 6. Real data.

$n = 225$  countries, two variables:  $X$ , the life expectancy at birth (years) in 2002 of the total population and  $Y$ , the difference between the life expectancy at birth of women and men. <http://www.odci.gov/cia/publications/factbook/>.

According to the PQD test proposed in (Scaillet, 2004), these data are PQD.



$$\hat{\rho}_{\text{NP}} = 52.4\%$$
$$\hat{\rho}_{\text{SP}} = 40.7\%$$



Estimation of high probability regions  $Q_\alpha$  from real data. Red:  $\alpha = 0.25$ , green:  $\alpha = 0.5$ , yellow:  $\alpha = 0.75$ . Top left: real data, top right: real data after rank transformation, bottom left: nonparametric estimate, bottom right: semiparametric estimate.

### Further work.

- Goodness of fit test.
- Study of the sub-family  $\phi(1) = 0$  without the assumption that  $\theta$  is a constant function.  
(what is the lower bound of  $\rho_{\theta,\phi}$ ?)
- Estimation of the function  $\theta$  in the general case.

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